Mock Exam Geometry - June 2018

Note: This exam consists of **four** problems. Usage of the theory and examples of Chapters 1:1-5, 2:1-5, 3:1-3, 4:1-6 of Do Carmo's textbook is allowed. You may not use the results of the exercises, *with the exception of* the results of Exercise 1-5:2,12, 4-3:1,2. Give a precise reference to the theory and/or exercises you use for solving the problems.

Problem 1 (10+10 = 20 pt.)
Consider the *twisted cubic* in $\mathbb{R}^3$, parametrized by $\alpha(t) = (t, t^2, t^3)$.

1. Calculate the curvature and torsion of the twisted cubic at an arbitrary point $\alpha(t)$.

2. Calculate the Frenet frame $\{\mathbf{t}(0), \mathbf{n}(0), \mathbf{b}(0)\}$ of the twisted cubic at the point $\alpha(0) = (0, 0, 0)$.

Problem 2 (10+10=20 pt.)
Let $g : \mathbb{R} \to \mathbb{R}$ be a C^∞-function, and let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = g(x^2 + y^2),$$

and let the surface S in $\mathbb{R}^3$ be the graph of f.

1. Prove that the point (x_0, y_0) is elliptic iff $\Delta(x_0, y_0) > 0$, and hyperbolic iff $\Delta(x_0, y_0) < 0$, where

$$\Delta(x, y) = g'(R)\left(g'(R) + 2Rg''(R)\right),$$

with $R = x^2 + y^2$.

2. Let $f(x, y) = x^2 + y^2 - (x^2 + y^2)^2$. Prove that the points of S that are neither elliptic nor hyperbolic form two circles, both consisting of parabolic points. (Note: in particular, S does not have any planar points.)

Problem 3 ($8 + 8 + 9 = 25$ pt.)
Let S be a regular surface in $\mathbb{R}^3$.

1. Prove: If S contains a line, then this line is an asymptotic curve of S.

In the remainder of this assignment S is a one-sheeted hyperboloid of revolution given by $x^2 + y^2 - z^2 = 1$. Furthermore, p is the point $(1, 0, 0)$ of S.

2. Show that the principal curvatures of S at p are equal to 1 and -1, and determine the curvature lines of S through p.

3. Determine both asymptotic directions of S at p, and the corresponding asymptotic curves of S through p.

Problem 4 $(8 + 8 + 9 = 25$ pt.)
Let C be a regular curve (without self-intersections) in the half-plane $\{(x, 0, z) \mid x > 0\}$. Let S be the surface of revolution in $\mathbb{R}^3$ obtained by rotating C about the z-axis.

1. Which meridians of S are geodesics? Give a proof of your statement(s).

2. Which parallel circles of S are geodesics? Give a proof of your statement(s).

The *angular momentum* of a regular curve $\alpha : \mathbb{R} \to S$ at $\alpha(t)$ is equal to $\alpha(t) \wedge \alpha'(t)$.

3. Prove that the z-component of the angular momentum of a geodesic on S is constant (i.e., independent of t).

Solutions

Problem 1.

1. We use the expressions for curvature and torsion from Exercise 1-5:12. To this end we compute

$$\alpha' = \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}, \quad \alpha'' = \begin{pmatrix} 0 \\ 2 \\ 6t \end{pmatrix}, \quad \alpha''' = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}, \text{ hence } \alpha' \wedge \alpha'' = \begin{pmatrix} 6t^2 \\ -6t \\ 2 \end{pmatrix}.$$

Therefore,

$$k(t) = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^{3/2}} = 2\sqrt{\frac{1+9t^2+9t^4}{(1+4t^2+9t^4)^3}},$$

$$\tau(t) = -\frac{\det(\alpha', \alpha'', \alpha''')}{|\alpha' \wedge \alpha''|^2} = -\frac{3}{1+9t^2+9t^4}.$$

2. We know that

$$\mathbf{t} = \frac{\alpha'}{|\alpha'|}, \quad \mathbf{b} = \frac{\alpha' \wedge \alpha''}{|\alpha' \wedge \alpha''|}, \quad \mathbf{n} = \mathbf{b} \wedge \mathbf{t}.$$

Therefore,

$$\mathbf{t}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{b}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{n}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Problem 2.

1. Use Example 6.5.4 of the first edition, which is Example 6.6.5 in the second edition: a point $(x, y, f(x, y))$ is elliptic (hyperbolic) iff $f_{xx}f_{yy} - f_{xy}^2$ is positive (negative) at the point (x, y). A straightforward computation yields

$$\begin{aligned} f_x &= 2xg'(R), \\ f_y &= 2yg'(R), \\ f_{xx} &= 2g'(R) + 4x^2g''(R), \\ f_{xy} &= 4xyg''(R), \\ f_{yy} &= 2g'(R) + 4y^2g''(R). \end{aligned}$$

The claim follows from $f_{xx}f_{yy} - f_{xy}^2 = 4g'(R)^2 + 8Rg'(R)g''(R) = 4\Delta(x, y)$.

2. In this particular case, $g(R) = R - R^2$, so $\Delta(x, y) = (1 - 2R)(1 - 6R)$. The set of points that are neither hyperbolic nor elliptic are the points on the graph of f with $R = x^2 + y^2 = \frac{1}{2}, z = g(R) = \frac{1}{4}$ or $R = x^2 + y^2 = \frac{1}{6}, z = g(R) = \frac{5}{36}$, which constitute two circles on S. These points are parabolic, and not planar, since at these points f has at least one non-zero second derivative.

Problem 3.

1. Let $\alpha(s)$ be a regular parametrization of the line, and let $\mathbf{v}$ be the direction vector of the line. Let $N(s)$ denote a differentiable unit normal vector field of S at $\alpha(s)$. Since the line lies on S, we have $\langle N(s), \mathbf{v}\rangle = 0$ for all s, so $\langle N'(s), \mathbf{v}\rangle = 0$. Therefore, the normal curvature at every point of the line in the direction $\mathbf{v}$ is zero, so the line is an asymptotic curve of S.

2. The lines of curvature of a surface of revolution are the meridians and the parallel circles. See [Do Carmo, Example 4 on page 161]. Therefore, the lines of curvature through p are the unit circle in the xy-plane (the parallel circle through p), and the branch of the hyperbola in the xz-plane with equation $x^2 - z^2 = 1, y = 0$ that contains p (the meridian through p).

The corresponding principal directions are e_2 and e_3, respectively (the second and third vector of the standard basis of $T_p\mathbb{R}^3$). Since S is given by $f(x, y, z) := x^2 + y^2 - z^2 - 1 = 0$, a unit normal field of S is

$$N(x, y, z) = \frac{\nabla f}{|\nabla f|}(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \begin{pmatrix} x \\ y \\ -z \end{pmatrix} = \frac{1}{\sqrt{1 + 2z^2}} \begin{pmatrix} x \\ y \\ -z \end{pmatrix}.$$

Note that $N(p) = e_1$ (the first vector of the standard basis of $T_p\mathbb{R}^3$). The parallel circle through p has parametrization $\alpha(t) = (\cos t, \sin t, 0)$, with $\alpha(0) = p$ and $\alpha'(0) = e_2$. The normal along this curve is

$$N(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}.$$

Therefore, $N'(0) = e_2$, so the normal curvature of S at p in the direction e_2 is equal to $k_n(e_2) = -\langle N'(0), e_2\rangle = -1$.

The meridian through p has parametrization $\beta(t) = (\sqrt{1 + t^2}, 0, t)$, with $\beta(0) = p$ and $\beta'(0) = e_3$. A similar computation shows that the normal curvature of this curve at p is equal to $k_n(e_3) = 1$.

Summarizing: the principal curvatures of S at p are -1 and 1, corresponding to the principal directions e_2 and e_3, respectively.

3. According to Euler's relation the asymptotic directions of S at p are given by $(\cos\vartheta)\, e_2 + (\sin\vartheta)\, e_3$, with $(-1)\cos^2\vartheta + (+1)\sin^2\vartheta = 0$, or, equivalently, $\tan^2\vartheta = 1$. Therefore $\vartheta = \pm\frac{\pi}{4}$, so the asymptotic directions of S at p are $e_2 \pm e_3$.

The lines through p with these asymptotic directions as direction vectors are parametrized by $s \mapsto (1, s, \pm s)$. These lines are on S, so they are the asymptotic curves of S through p (cf Part 1).

Problem 4.

Let the curve in the xz-plane be parametrized by $s \mapsto (f(s), 0, g(s))$, with $f(s) > 0$ for all s in the parameter interval I. Since the curve is regular, we may assume that is has unit speed, i.e.,

$$f'(s)^2 + g'(s)^2 = 1.$$

A parametrization of the surface is given by

$$\mathbf{x}(u, v) = (f(v)\cos u, f(v)\sin u, g(v)),$$

with $(u, v) \in U \wedge I$. Here U is a suitable parameter domain for the angular variable u. A simple computation (see Example 4 on page 161) shows that the coefficients of the first fundamental form are given by

$$E(u, v) = f(v)^2, \quad F(u, v) = 0, \quad G(u, v) = 1.$$

1. Since $G = 1$, a meridian of S has unit speed parametrization $\alpha(s) = \mathbf{x}(u_0, s))$, for some fixed $u_0 \in \mathbb{R}$. We claim that its acceleration vector $\alpha''(s)$ is perpendicular to S at every point $\alpha(s)$. To see this, observe that $\alpha''(s) = \mathbf{x}_{vv}(u_0, s)$, and

$$\langle \mathbf{x}_{vv}, \mathbf{x}_u \rangle = F_v - \tfrac{1}{2} G_u = 0, \quad \text{and} \quad \langle \mathbf{x}_{vv}, \mathbf{x}_v \rangle = \tfrac{1}{2} G_v = 0.$$

Therefore, $\alpha''(s)$ is perpendicular to $T_{\alpha(s)}S$ for all s. Therefore, every meridian is a geodesic of S.

2. Since E is constant along a parallel circle $v = v_0$ of S, such a parallel circle has unit speed parametrization $\beta(s) = \mathbf{x}(\omega_0 s, v_0)$, for a suitably chosen $\omega_0 > 0$. More precisely, $\beta'(s) = \omega_0 \mathbf{x}_u(\omega_0 s, v_0)$, so

$$|\beta'(s)| = \omega_0 \sqrt{E(\omega_0 s, v_0)} = \omega_0 f(v_0),$$

so we take $\omega_0 = f(v_0)^{-1}$. We look for parallels with $\beta''(s)$ perpendicular to S for all s. Since $\beta''(s) = \omega_0^2 \mathbf{x}_{uu}(\omega_0 s, v_0)$, and

$$\langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle = \tfrac{1}{2} E_u = 0, \quad \text{and} \quad \langle \mathbf{x}_{uu}, \mathbf{x}_v \rangle = F_u - \tfrac{1}{2} E_v = -ff',$$

we see that

$$\langle \beta''(s), \mathbf{x}_u(\omega_0 s, v_0) \rangle = 0,$$

and

$$\langle \beta''(s), \mathbf{x}_v(\omega_0 s, v_0) \rangle = -\omega_0^2 f(v_0) f'(v_0) = -\omega_0 f'(v_0).$$

Therefore, the parallel circle is a geodesic iff $f'(v_0) = 0$ (since $\omega_0 > 0$), i.e., iff the contour curve C has a vertical tangent at $(f(v_0), 0, g(v_0))$. Equivalently, in this case the meridian through every point of the parallel circle has a vertical tangent.

3. Let $\gamma(t) = \mathbf{x}(u(t), v(t))$ be a curve on S, then

$$\langle \gamma \wedge \gamma', e_3 \rangle = u' \langle \mathbf{x} \wedge \mathbf{x}_u, e_3 \rangle + v' \langle \mathbf{x} \wedge \mathbf{x}_v, e_3 \rangle,$$

where u', v', γ and γ' are evaluated at t, and $\mathbf{x}, \mathbf{x}_u$ and $\mathbf{x}_v$ are evaluated at $(u(t), v(t))$. Let $\xi(u, v) = f(v)\cos u$ and $\eta(u, v) = f(v)\sin u$ be the x- and the y-component of $\mathbf{x}(u, v)$, respectively. Then

$$\langle \mathbf{x} \wedge \mathbf{x}_u, e_3 \rangle = \begin{vmatrix} \xi & \xi_u \\ \eta & \eta_u \end{vmatrix} = \begin{vmatrix} f(v)\cos u & -f(v)\sin u \\ f(v)\sin u & f(v)\cos u \end{vmatrix} = f(v)^2.$$

A similar computation shows that $\langle \mathbf{x} \wedge \mathbf{x}_v, e_3 \rangle = 0$. Therefore,

$$\langle \gamma \wedge \gamma', e_3 \rangle = f(v(t))^2 \, u'(t).$$

If γ is a geodesic, then $f^2 u'$ is constant along γ. See Clairaut's relation on page 256. In other words, the angular momentum is constant along geodesics.